On unitary group representations in spaces with indefinite metric

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1. Let G be a topological group and X a linear topological space. By a representation of G in X there is meant a mapping $g \to U_g$ of G into linear continuous operators U_g in X satisfying the following conditions:

1. $U_e = 1$, where e is the group unit in G and 1 is the identity operator in X;
2. $U_{g_1 g_2} = U_{g_1} U_{g_2}$ for any $g_1, g_2 \in G$;
3. $g \to U_g x$ is a continuous mapping of G into X for any fixed $x \in X$.

X is then called the *representation space.*

A representation $g \to U_g$ is called *irreducible* if no non-trivial closed subspace of X exists, which is invariant with respect to all operators U_g, $g \in G$. A representation $g \to U_g$ is called *unitary,* if X is a Hilbert space with the norm topology and all U_g are unitary operators in X.

In what follows we consider only locally compact groups with a countable neighborhood basis. It is known (cf. e. g. [1] ch. 8) that every unitary representation of such a group in a separable Hilbert space can be realized as a direct integral of irreducible representations. For non-unitary representations no analogous result holds; even in finite dimensional spaces there can exist reducible but not completely reducible (i. e. non decomposable) representations.

So a question arises: in what manner can a reducible representation be constructed with the aid of irreducible ones. There is no hope, at the present stage of the spectral theory of non self-adjoint operators, to solve this problem in its full generality. It seems, however, to be reasonable to develop a theory of sufficiently large classes of non-unitary group representations, which would be similar to the theory of generalized spectral operators with spectral singularities constructed by LYANZE [2—5] and having its origin in DUNFORD's [6] theory of spectral operators and in my works [7, 8] on spectral decomposition of non-self adjoint differential operators.

As the first step in the realization of this program we consider representations in Pontryagin spaces $\Pi_\varkappa$, which are unitary in the indefinite metric of $\Pi_\varkappa$, but not unitary in the usual sense.

The results which will be reported here, are to be considered as the beginning of the corresponding theory. We recall that the space $\Pi_\varkappa$ can be defined in the following manner. It is a Hilbert space H with a usual inner product $[x, y]$ and an indefinite inner product (x, y) which, for some complete orthonormal system $\{e_j\}$ in H, is defined by

$$(x, y) = \sum_{j=1}^{\varkappa} \xi_j \bar{\eta}_j - \sum_{j>\varkappa} \xi_j \bar{\eta}_j,$$

where

$$\xi_j = [x, e_j], \quad \eta_j = [y, e_j],$$

$\varkappa$ is a fixed positive integer and $\varkappa < \dim H$. A set $S \subset \Pi_\varkappa$ is called positive (resp. negative, non-negative, non-positive, null) if $(x, x) > 0$ (resp. $< 0, \geqq 0, \leqq 0, = 0$) for $x \in S, x \neq 0$.

A linear operator U in $\Pi_\varkappa$ is called *unitary* in $\Pi_\varkappa$ if it maps $\Pi_\varkappa$ onto $\Pi_\varkappa$ in one-to-one manner and $(Ux, Uy) = (x, y)$ for all $x, y \in \Pi_\varkappa$. A representation $g \to U_g$ in $\Pi_\varkappa$ is called *unitary in* $\Pi_\varkappa$ if all U_g are unitary in $\Pi_\varkappa$. The study of such representations must be based on the theory of operator algebras B in $\Pi_\varkappa$ which are symmetric in the following sense: if $A \in B$, then also $A^* \in B$, where A^* is defined by

$$(Ax, y) = (x, A^* y). \tag{1}$$

The study of such general algebras is not begun and many problems about such algebras remain open. But if we want, for the case of unitary representations in $\Pi_\varkappa$, to solve the problem of description of the reducible representations posed above, then only *commutative* symmetric operator algebras in $\Pi_\varkappa$ need be considered.

The theory of such algebras is closely related, but not reduces to the spectral theory of J-self-adjoint operators in $\Pi_\varkappa$ spaces developed by M. Krein and H. Langer [9].

2. Let B and B' be two symmetric commutative operator algebras in spaces $\Pi_\varkappa$ and $\Pi'_\varkappa$; B and B' will be called equivalent if a linear mapping of $\Pi_\varkappa$ onto $\Pi'_\varkappa$ exists, which preserves the indefinite inner product and maps B onto B'. A problem arises to describe commutative symmetric operator algebras in $\Pi_\varkappa$ up to equivalence. The solution of an analogous problem is known for operator algebras in usual Hilbert space (cf. e. g. [10] or [1]); we give it here for the space $\Pi_\varkappa$. The main tool is the following result [11], which is a generalization to operator families of the Pontryagin—Krein—Johvidov theorem (See e. g. [12]).

Theorem 1. *For every family of commuting unitary operators in $\Pi_\varkappa$ there exists a non-negative $\varkappa$-dimensional subspace which is invariant with respect to all operators of the family.* [1])

Now let B be a commutative symmetric operator algebra in $\Pi_\varkappa$. As a corollary to Theorem I, it follows that a non-negative $\varkappa$-dimensional subspace $\mathfrak{P}$ exists, which is invariant with respect to all $A \in B$. But $\mathfrak{P}$ being finite-dimensional, a vector $x \in \mathfrak{P}$, $x \neq 0$, must exist which is a common eigenvector for all $A \in B$, i. e.

$$Ax = \lambda(A)x \text{ for all } A \in B. \tag{2}$$

The function $A \to \lambda(A)$ is easily seen to be a homomorphism of B into the complex number field C. Any such homomorphism $A \to \lambda(A)$ will be called an eigenfunctional of B if a non-negative vector $x \neq 0$ exists, satisfying condition (2). The argument above shows, that eigenfunctionals always exist. For an eigenfunctional $\lambda(A)$ the set

$$S_\lambda = \{x : x \in \Pi_\varkappa, (A - \lambda(A) \cdot 1)^k x = 0 \text{ for some } k = k(x) \text{ and all } A \in B\}$$

[1]) The assertion of Theorem I remains also valid for a family of non-commuting unitary operators in $\Pi_\varkappa$, if the family forms a unitary representation of a connected solvable group (see [13]). We note also that Theorem I contains the solution for $\Pi_\varkappa$ of a problem posed by Phillips [14].

is called the *root manifold* of λ. It is easily seen to be invariant with respect to all operators $A \in B$.

An eigenfunctional $\lambda(A)$ of B will be called *real*, if $\lambda(A^*) = \overline{\lambda(A)}$ for all $A \in B$, and *non-real* in the contrary case. The non-real eigenfunctionals of B, if they exist at all, form a finite set $\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots, \lambda_\sigma, \mu_\sigma$ of distinct functionals, where $\mu_j(A) = \lambda_j(A^*)$.

The corresponding root manifolds S_{λ_j}, S_{μ_j} are skewly related nullspaces. This means that no vector $x \in S_{\lambda_j}$, $x \neq 0$, is orthogonal to S_{μ_j} and no vector $y \in S_{\mu_j}$, $y \neq 0$, is orthogonal to S_{λ_j}. Thus $S_{\lambda_j} \dotplus S_{\mu_j}$ is a finite-dimensional $\Pi_\varkappa$ space, which is invariant with respect to all $A \in B$ and the restriction of B to $S_{\lambda_j} \dotplus S_{\mu_j}$ can be easily described. Moreover, $S_{\lambda_j} \dotplus S_{\mu_j} \perp S_{\lambda_k} \dotplus S_{\mu_k}$ for $j \neq k$ with respect to (x, y). The space $H = \sum_{j=1}^{\sigma} \oplus (S_{\lambda_j} \dotplus S_{\mu_j})$ is called the *hyperbolic space* of B. It follows from the above that H is invariant with respect to all $A \in B$, hence $H^\perp$ has the same property. The restriction of B to H is easily described and the restriction of B to $H^\perp$ has no non-real eigenfunctionals. So in what follows we may assume that B has no non-real eigenfunctionals. Let $\mathfrak{P}$ be a non-negative $\varkappa$-dimensional subspace, invariant with respect to all $A \in B$, and let $\lambda_1, \ldots, \lambda_p$ be all the distinct (real) eigenfunctionals of B with eigenvectors in $\mathfrak{P}$. Put $\varrho_j = \dim (S_{\lambda_j} \cap \mathfrak{P})$. Then λ_j and ϱ_j do not depend on the choice of $\mathfrak{P}$ and the λ_j form the set of all distinct (real) eigenfunctionals of B. We put

$$\mathfrak{Q}_j = \{x : (A - \lambda(A)1)^{\varrho_j} x = 0 \text{ for all } A \in B\} \tag{3}$$

and

$$\mathfrak{Q} = \sum_{j=1}^{p} \cdot \mathfrak{Q}_j, \tag{4}$$

then $\mathfrak{Q}$ is a closed subspace in $\Pi_\varkappa$ and any $\varkappa$-dimensional non-negative subspace, which is invariant with respect to all $A \in B$, is contained in $\mathfrak{Q}$.

It follows that the space $\mathfrak{M} = \mathfrak{Q}^\perp$ is non-positive. The spaces $\mathfrak{Q}$ and $\mathfrak{M}$ are called *the principal space* and *the basic space* of B. The intersection $\mathfrak{N} = \mathfrak{Q} \cap \mathfrak{M}$ is a nullspace; it is called *the basic nullspace* of B. All these subspaces are easily seen to be invariant with respect to all $A \in B$.

Now let $\mathfrak{N}'$ be a nullspace in $\Pi_\varkappa$ which is skewly related to $\mathfrak{N}$; we put $\mathfrak{H} = \mathfrak{M} \cap \mathfrak{N}'^\perp$, $\Pi = \mathfrak{Q} \cap \mathfrak{N}'^\perp$. Then $\mathfrak{H}$ is negative, i. e. essentially a Hilbert space, Π positive, negative, or a $\Pi_\varkappa$-space, and

$$\mathfrak{M} = \mathfrak{N} \oplus \mathfrak{H}, \tag{5}$$

$$\mathfrak{Q} = \mathfrak{N} \oplus \Pi, \tag{6}$$

$$\Pi_\varkappa = (\mathfrak{N} \dotplus \mathfrak{N}') \oplus \mathfrak{H} \oplus \Pi. \tag{7}$$

We consider first B on $\mathfrak{N}$. As $\mathfrak{N}$ is finite-dimensional (as a nullspace), and all $A \in B$ commute, there exists in $\mathfrak{N}$ a basis

$$\{x_{jl}\} \qquad \left(l = 1, \ldots, r_j;\ j = 1, \ldots, q;\ \sum_{j=1}^{q} r_j = \dim \mathfrak{N} \right)$$

such that the matrix of any $A \in B$ in this basis has a triangular form, namely

$$(8) \qquad Ax_{jl} = \sum_{s=1}^{l} \lambda_{jls}(A) x_{js}$$

with diagonal elements $\lambda_{jll}(A) = \lambda_j(A)$, $j = 1, \ldots, q$.

Let $\{y_{jl}\}$ be a basis in $\mathfrak{N}'$, which is biorthogonal to $\{x_{jl}\}$.

Using (7) we put

$$(9) \qquad A^* y_{jl} = n_{jl}(A) + n'_{jl}(A) + h_{jl}(A) + \pi_{jl}(A),$$

where

$$n_{jl}(A) \in \mathfrak{N}, \quad n'_{jl}(A) \in \mathfrak{N}', \quad h_{jl}(A) \in \mathfrak{H}, \quad \pi_{jl}(A) \in \Pi.$$

With the use of (5) and (9) we get further

$$(10) \qquad Ah = \sum_{j=1}^{q} \sum_{l=1}^{r_j} (h, h_{jl}(A)) x_{jl} + A_1 h \quad \text{for} \quad h \in \mathfrak{H},$$

where A_1 is an operator in $\mathfrak{H}$. The mapping $A \to A_1$ is a norm-continuous symmetric homomorphism of B onto a commutative symmetric operator algebra B_1 in the Hilbert space $\mathfrak{H}$, with A_1^* the usual adjoint operator.

The algebra B_1 does not depend, up to equivalence, on the choice of $\mathfrak{N}'$. We suppose that B contains the identity operator 1 and that $\Pi_\varkappa$ is separable; then B_1 can be realized in the following known manner (see e. g. [10] or [1]). Let $\bar{B}_1$ be the closure of B_1 in the operator norm, T the bicompact space of maximal ideals t of $\bar{B}_1$, and $A(t)$ the value of $A_1 \in B_1$ at t. There exists a Borel measure σ on T and a σ-measurable family of Hilbert spaces $\mathfrak{H}(t)$, $t \in T$, such that, up to equivalence,

$$(11) \qquad \mathfrak{H} = \int_T \mathfrak{H}(t)\, d\sigma,$$

$$(12) \qquad A_1\{h(t)\} = \{A(t) h(t)\}$$

for $h = \{h(t)\} \in \mathfrak{H}$, $A_1 \in B_1$, $\{A(t): A_1 \in \bar{B}_1\} = C(T)$, and $\{A(t): A_1 \in B_1\}$ is dense in $C(T)$. In an analogous manner we have

$$(13) \qquad A\pi = \sum_{j=1}^{q} \sum_{l=1}^{r_j} (\pi, \pi_{jl}(A)) x_{jl} + A_2 \pi \quad \text{for} \quad \pi \in \Pi,$$

where A_2 is an operator in Π. The mapping $A \to A_2$ is also a continuous symmetric homomorphism onto a symmetric commutative operator algebra B_2 in Π, which up to equivalence does not depend on the choice of $\mathfrak{N}'$. Using further (3) and (4) we find that $\Pi = \sum_{j=1}^{p} \oplus \Pi^j$, where Π^j are subspaces of Π, invariant with respect to all $A_2 \in B_2$. Moreover, on each Π^j we have

$$(14) \qquad (A_2 - \lambda_j(A_2) 1)^{\varrho_j} = 0 \quad \text{for all } A_2 \in B_2.$$

An algebra satisfying (14) will be called *degenerate*. So we see that the restriction of B_2 on each Π^j is a degenerate algebra. If Π^j is positive or negative, then the degenerate algebra is easily seen to be the algebra of the scalar multiples of the identity operator. If Π^j is a Π_κ space, then the general form of degenerate algebras can be described by using Theorem I. Finally the $n_{jl}(A)$ and $n'_{jl}(A)$ have the form

$$n_{jl}(A) = \sum_{j'=1}^{q} \sum_{l'=1}^{r_{j'}} \alpha_{jlj'l'}(A) x_{j'l'}, \tag{15}$$

$$n'_{jl}(A) = \sum_{l'=l}^{r_j} \overline{\lambda_{jl'l}(A)} y_{jl'}, \tag{16}$$

where $\alpha_{jlj'l'}(A)$ are some complex valued functions on B. Substituting this in (9) with A^* instead of A we get

$$Ay_{jl} = \sum_{j'=1}^{q} \sum_{l'=1}^{r_{j'}} \alpha_{jlj'l'}(A^*) x_{j'l'} + \sum_{l'=l}^{r_j} \overline{\lambda_{jl'l}(A^*)} y_{jl'} + h_{jl}(A^*) + \pi_{jl}(A^*). \tag{17}$$

Equations (8), (10), (13), (17) define the operators $A \in B$. If we want now to describe up to equivalence all possible commutative symmetric algebras B, we omit A in the right hand and write these equations in the form:

$$Ax_{jl} = \sum_{s=1}^{l} \lambda_{jls} x_{js}, \tag{18}$$

$$Ah = \sum_{j=1}^{q} \sum_{l=1}^{r_j} (h, h_{jl}) x_{jl} + A_1 h, \tag{19}$$

$$A\pi = \sum_{j=1}^{q} \sum_{l=1}^{r_j} (\pi, \pi_{jl}) x_{jl} + A_2 \pi, \tag{20}$$

$$Ay_{jl} = \sum_{j'=1}^{q} \sum_{l'=1}^{r_{j'}} \alpha^*_{jlj'l'} x_{j'l'} + \sum_{l'=l}^{r_j} \overline{\lambda^*_{jl'l}} y_{jl'} + h^*_{jl} + \pi^*_{jl}, \tag{21}$$

where the systems $\xi = \{\lambda_{jls}, \alpha_{jlj'l'}, h_{jl}, \pi_{jl}, A_1, A_2\}$ defining the operators A run over some linear manifold Ξ on which an involution $\xi \to \xi^* = \{\lambda^*_{jls}, \alpha^*_{jlj'l'}, h^*_{jl}, \pi_{jl}\, A^*_1, A^*_2\}$ is defined and which satisfies the following set of axioms, expressing the fact, that the corresponding operators A run over a commutative symmetric algebra:

1. $\xi^{**} = \xi$;
2. $(c\xi)^* = \bar{c}\xi^*$ for $c \in C, \xi \in \Xi$;
3. $(\xi + \xi')^* = \xi^* + \xi'^*$ for $\xi, \xi' \in \Xi$;
4. A^*_1, A^*_2 coincide with the adjoint operators in B_1, B_2, $\alpha^*_{jlj'l'} = \bar{\alpha}_{j'l'jl}$;

5. for any $\xi, \xi' \in \Xi$ we have

$$\sum_{s'=s}^{l} \lambda_{jls'} \lambda'_{js's} = \sum_{s'=s}^{l} \lambda'_{jls'} \lambda_{js's}, \tag{22}$$

$$\sum_{l=s}^{r_j} \overline{\lambda}_{jls} h'_{jl} + A_1'^{*} h_{js} = \sum_{l=s}^{r_j} \overline{\lambda}'_{jls} h_{jl} + A_1^{*} h'_{js}, \tag{23}$$

$$\sum_{l=s}^{r_j} \overline{\lambda}_{jls} \pi'_{jl} + A_2'^{*} \pi_{js} = \sum_{l=s}^{r_j} \overline{\lambda}'_{jls} \pi_{jl} + A_2^{*} \pi'_{js}, \tag{24}$$

$$\sum_{l'=s}^{r_{j'}} \alpha_{jlj'l'} \lambda'^{*}_{j'l's} + \sum_{l'=l}^{r_j} \overline{\lambda}_{jl'l} \alpha'_{jl'j's} + (h_{jl}, h'^{*}_{j's}) + (\pi_{jl}, \pi'^{*}_{j's}) = \tag{25}$$

$$= \sum_{l'=s}^{r_{j'}} \alpha'_{jlj'l'} \lambda^{*}_{j'l's} + \sum_{l'=l}^{r_j} \lambda'_{jl'l} \alpha_{jl'j's} + (h'_{jl}, h^{*}_{j's}) + (\pi'_{jl}, \pi^{*}_{j's});$$

6. if $\xi, \xi' \in \Xi$ then also $\xi'' \in \Xi$, where

$$A_1'' = A_1 A_1', \quad A_2'' = A_2 A_2', \quad \lambda''_{jls} = \sum_{s'=s}^{l} \lambda_{jls'} \lambda'_{js's},$$

$$\alpha''_{jlj s} = \sum_{l'=s}^{r_{j'}} \alpha_{jlj'l'} \lambda'^{*}_{j'l's} + \sum_{l'=l}^{r_j} \overline{\lambda}_{jl'l} \alpha'_{jl'j's} + (h_{jl}, h'^{*}_{j's}) + (\pi_{jl}, \pi'^{*}_{j's}),$$

$$h''_{js} = \sum_{l=s}^{r_j} \overline{\lambda}_{jls} h'_{jl} + A_1'^{*} h_{js}, \quad \pi''_{js} = \sum_{l=s}^{r_j} \overline{\lambda}_{jls} \pi'_{jl} + A_2'^{*} \pi_{js}.$$

If we take all such systems Ξ we obtain by (18)—(21), up to equivalence, all possible commutative symmetric operator algebras in $\Pi_\varkappa$. Now suppose that B is separable in the operator norm. Then using one of these axioms namely (23) and the realization (11)—(12) of $\mathfrak{H}$ and B_1 a formula for $h_{jl}(A) = h_j(A, t)$ can be obtained. In order to write this formula we need some definitions. An eigenfunctional λ_j is called *singular* if a point $t_j \in T$ exists, such that $\lambda_j(A) = A(t_j)$ for all $A \in B$; t_j is then called *the corresponding singular point* of B. If no such point exists λ_j is called *regular*. The singular points are analogous to the spectral singularities of non selfadjoint operators (cf. [2—5]).

We put: $T_j = T - \{t_j\}$ if λ_j is a singular functional and $T_j = T$ otherwise, $K_j = \mathfrak{H}(t_j)$ if λ_j is singular and $\sigma(\{t_j\}) > 0$ and $K_j = (0)$ otherwise. K_j is a Hilbert space which we call *the singular space of B corresponding to λ_j*. Then we have

$$h_{jl}(A, t) = \left(\overline{A(t)} - \overline{\lambda_j(A)}\right) \zeta_{jl}(t) - \sum_{\mu=l+1}^{r_j} \overline{\lambda_{j\mu l}(A)} \zeta_{j\mu}(t) \quad \text{for } t \neq t_j \tag{26a}$$

$$h_{jl}(A, t_j) = k_{jl}(A) \in K_j \tag{26b}$$

where $\zeta_{jl}(t) \in \mathfrak{H}(t)$, $\zeta_{jl}(t)$ is a σ-measurable function of t such that the right hand side of (26a) belongs to $\mathfrak{H}$ for any $A_1 \in B_1$. Formula (19) can now be written as follows

$$(27) \qquad A\{h(t)\} = -\sum_{j=1}^{q}\sum_{l=1}^{r_j}\left\{\int_{T_j}[h(t),\, h_{jl}(t)]\,d\sigma + [k,\, k_{jl}]\right\} x_{jl} + \{A(t)h(t)\}$$

where $h_{jl}(t) = h_{jl}(A, t)$ is given by (26a).

Equations (18), (27), (20), (21) with $\xi = \{\lambda_{jls}, \alpha_{jlj'l'}, h_{jl}, \pi_{jl}, A_1, A_2\}$ running over some Ξ satisfying 1—6 give the general model of all symmetric commutative separable algebras B in $\Pi_\varkappa$.

We remark that conditions can be given in order that two such models be equivalent [15].

We also remark that the above construction can be simplified if we take for $\mathfrak{Q}$ not the principal space, but any $\varkappa$-dimensional non-negative subspace, which is invariant with respect to all $A \in B$. Then the corresponding Π is finite dimensional and positive and the corresponding degenerate algebras are algebras of scalars. This modified construction can be used to describe degenerate algebras in $\Pi_\varkappa$ [16]. But this construction will depend on the choice of $\mathfrak{Q}$ and moreover it will be applicable to group representations for commutative groups only while the construction with the principal space for $\mathfrak{Q}$ is applicable to group representation of non-commutative groups as well.

For $\varkappa = 1$ the above construction can be simplified ([17, 18]).

3. Let us now turn to group representations. Let $g \to U_g$ be a unitary representation of G in a separable $\Pi_\varkappa$. We denote by M the set of all bounded linear operators in $\Pi_\varkappa$ which commute with all $U_g, g \in G$ and by $\mathfrak{A}$ a maximal commutative subalgebra of M. As $\Pi_\varkappa$ is separable there exists a sequence $A^{(n)} \in \mathfrak{A}$ which is dense in $\mathfrak{A}$ in the strong operator topology. Let B be the symmetric subalgebra of $\mathfrak{A}$, closed with respect to the operator norm convergence, which is generated by the $A^{(n)}$ and 1. We apply to this B the above results. Let $H = \sum_{j=1}^{\sigma} \oplus (S_{\lambda_j} \dotplus S_{\mu_j})$ be the hyperbolic space of B. Then each S_{λ_j}, S_{μ_j} is invariant with respect to all U_g and the restrictions of the representation $g \to U_g$ to S_{λ_j} and S_{μ_j} are operatorly irreducible, finite dimensional representations, adjoint one to another. We recall that a representation $g \to V_g$ is called *operatorly irreducible*, if any bounded linear operator in the representation space, which commutes with all V_g, is a multiple of the identity operator. The orthogonal complement $H^\perp$ is also invariant with respect to all U_g and considering the representation on $H^\perp$ we may further assume that $H = (0)$. The principal space $\mathfrak{Q}$, the basic space $\mathfrak{M}$, and the basic nullspace $\mathfrak{N}$ of B are all invariant with respect to all $U_g, g \in G$. Let $\mathfrak{N}', \mathfrak{H}, \Pi, \{x_{jl}\}, \{y_{jl}\}, \mathfrak{H}(t), K_j, t_j, \zeta_{jl}(t)$ be introduced as before.

Then up to equivalence we may assume that $\mathfrak{H} = \int_T \mathfrak{H}(t)\,d\sigma$ and we have the following:

Theorem 2. *Let $g \to U_g$ be a unitary representation of G in a separable space $\Pi_\varkappa$ and let B be the corresponding commutative symmetric algebra in $\Pi_\varkappa$, constructed above. To the realization of B as a model there corresponds a realization of the representation*

sentation $g \to U_g$ *by the equations*

(28) $$U_g x_{jl} = \sum_{s=1}^{r_j} u_{jls}(g) x_{js},$$

(29) $$U_g\{h(t)\} = \sum_{j=1}^{q}\sum_{l=1}^{r_j} (h, h_{jl}(g)) x_{jl} + \{U_g(t) h(t)\},$$

(30) $$U_g \pi = \sum_{j=1}^{q}\sum_{l=1}^{r_j} (\pi, \pi_{jl}(g)) x_{jl} + U_g^{[1]}\pi,$$

(31) $$U_g y_{jl} = \sum_{j'=1}^{q}\sum_{l'=1}^{r_{j'}} \alpha_{jlj'l'}(g) x_{j'l'} + \sum_{l'=1}^{r_j} \overline{u_{jl'l}(g^{-1})} y_{jl} + h_{jl}(g^{-1}) + \pi_{jl}(g^{-1}),$$

where $g \to U_g(t)$ *is an irreducible unitary representation of* G *in* $\mathfrak{H}(t)$ *for* σ*-almost every* $t \in T$, $t \neq t_j$, σ*-measurably depending on* t, *and* $g \to U_g^{[1]}$ *is a unitary representation of* G *in* Π. $u_{jls}(g)$, $\alpha_{jlj'l'}(g)$, $h_{jl}(g)$, $\pi_{jl}(g)$ *are continuous functions on* G *with values from* C, C, $\mathfrak{H}$, Π, *respectively*; $h_{jl}(t) = U_{g^{-1}}(t)\zeta_{jl}(t) - \sum_{\mu}^{r_j}{}' \overline{u_{j\mu l}(g)}\zeta_{j\mu}(t)$ *for* $t \neq t_j$,

and $$h_{jl}(t_j) = k_{jl}(g) \in K_j.$$

Remark. It is natural to expect, that for particular classes of groups more detailed results could be obtained. In this direction little is done and only representations of the 2×2 complex unimodular group $SL(2, C)$ (which is essentially the Lorentz group) were discussed (cf. [19]—[22]).

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